Heat conduction in relativistic neutral gases revisited

A. L. García-Perciante , A. R. Méndez

Depto. de Matemáticas Aplicadas y Sistemas,

Universidad Autónoma Metropolitana-Cuajimalpa,

Artificios 40 México D.F 01120, México

Abstract

The kinetic theory of dilute gases to first order in the gradients yields linear relations between forces and fluxes. The heat flux for the relativistic gas has been shown to be related not only to the temperature gradient but also to the density gradient in the representation where number density, temperature and hydrodynamic velocity are the independent state variables. In this work we show the calculation of the corresponding transport coefficients from the full Boltzmann equation and compare the magnitude of the relativistic correction.

I. INTRODUCTION

It has been recently shown that, when constitutive equations to first order in the gradients are introduced in the transport equations for a relativistic gas, the system presents no intsability nor causality issues [1]. Moreover, the pathology identified by Hiscock and Lindblom [2], which in part lead to ruling out first order theories, is due to the coupling of heat with acceleration. This coupling has a phenomenological origin and is in contradiction with the results obtained from relativistic kinetic theory. Due to this fact, the first order theories are being currently reexamined and proposed as solid frameworks from where one could extract the physics of high temperature systems present both in astrophysical and experimental scenarios [1].

In this paper, the heat flux constitutive equation is obtained together with the associated transport coefficients using the full collision kernel in Boltzmann equation to first order in the gradients using a representation where the density, hydrodynamic velocity and temperature are the independent state variables. The heat flux in this scenario is coupled with temperature and density gradients and two transport coefficients are identified. The explicit form of such coeficients are obtained for a constant scattering cross section model and the results are shown to be consistent with the ones obtained by other authors using a different representation *only in the comoving frame*. However, for an arbitrary observer, the stress energy tensor includes Lorentz transformation factors [3]. The form of such tensor, from which the transport equations are to be extracted, is briefly discussed.

The present work is divided as follows. In Section II the theoretical framework that sustains the calculation is presented. In Section III the mathematical problem is set up as two separate integro-differential equations whose solutions are formulated as expansions in orthogonal polynomials from which the general form for the transport coefficients is obtained. A constant scattering section model is assumed in Section IV in order to calculate the collision integrals and compare the transport coefficients as functions of z. The discussion of the results, including their comparison with the solution obtained in Ref. [4] as well as final remarks regarding the form of the stress-energy tensor are included in Section V.

II. SPECIAL RELATIVISTIC BOLTZMANN EQUATION

Consider a neutral, dilute, single component, non-degenerate relativistic fluid in the absence of external fields. Additionally, assume that such a gas is characterized by values of the relativistic parameter z, defined as the ratio of the thermal energy to the rest energy a single particle, close to one. Much higher values of this parameter correspond to very high temperature, ultra-relativistic gases for which the neutrality of the particles here assumed may not be appropriate. The non-relativistic gas corresponds to the limit where z tends to zero. Thus, for the system here considered, the space-time is given by a Minkowski metric with a $+++-$ signature for which the position and velocity four-vectors are given by

$$x^{\nu} = (\vec{x}, ct) \qquad v^{\nu} = \gamma_{(w)} (\vec{w}, c) \tag{1}$$

where c is the speed of light and $\gamma_{(w)} = \left(1 - w^2/c^2\right)^{-1/2}$.

The distribution function f is an invariant such that $f\left(x^{\nu}, v^{\nu}\right) d^3x d^3v$, also an invariant, is the number of particles contained in a volume in phase space. The relativistic Boltzmann equation for the evolution of f reads [4–6],

$$v^{\alpha} f_{,\alpha} = J\left(f, f'\right) \tag{2}$$

where the contraction on the left side corresponds to a total proper-time derivative. That is, in the molecule's rest frame where $\vec{w} = \vec{0}$

$$v^{\alpha} f_{,\alpha} = \frac{v^4}{c} \frac{\partial f}{\partial \tau} = \frac{df}{d\tau} \tag{3}$$

The right hand side of Eq. (2) is given by

$$J\left(f f'\right) = \int \int \left\{f' f_1{}' - f f_1\right\} \mathcal{F} \sigma\left(\Omega\right) d\Omega dv_1^* \tag{4}$$

where $dv_1^* = d^3v_1/v_1^4$ and $\mathcal{F}$ is the invariant flux given by [4]

$$\mathcal{F} = \frac{1}{c^2} v^4 v_1^4 = \frac{1}{c} \sqrt{\left(v^{\alpha} v_{1\alpha}\right)^2 - c^4} = \frac{1}{c} \sqrt{\left(\gamma_{(w)} \gamma_{(w_1)} \left(\vec{w} \cdot \vec{w_1} - c^2\right)\right)^2 - c^4} \tag{5}$$

which reduces to the relative velocity in the non-relativistic limit. The solution to the homogeneous Boltzmann equation is determined by $J\left(f f'\right) = 0$ together with the requirement of consistency with the local equilibrium assumption. That is, thermodynamic equilibrium is locally assumed and thus the state variables are given by

$$N^{\nu} = \int f^{(0)} v^{\nu} dv^* \tag{6}$$

$$\mathcal{T}^{\mu\nu} = \int f^{(0)} v^{\mu} v^{\nu} d^{*}v \tag{7}$$

which are the particle flux and equilibrium stress-energy tensor respectively. The thermodynamical, local equilibrium, variables for the system can then be extracted from the previous tensors as

$$n = -\frac{N^{\nu}\mathcal{U}_{\nu}}{c^{2}} \tag{8}$$

$$n\varepsilon = \frac{\mathcal{U}_{\mu}\mathcal{U}_{\nu}}{c^{2}}\mathcal{T}^{\mu\nu}. \tag{9}$$

It has been recently shown that the heat flux can be defined, as in the non-relativistic case, as the average of the peculiar or chaotic kinetic energy [3]. Thus, since this velocity is the one measured by an observer locally comoving with the fluid element, the calculation will be performed in a comoving frame. Therefore, the fluid's hydrodynamic four velocity has only a temporal component:

$$\mathcal{U}^{\nu} = \left(\vec{0},\, c\right) \tag{10}$$

The dissipative terms in the stress-energy tensor, as defined in Ref. [3] from a standard tensor decomposition [7], include a four-vector that in an arbitrary frame can be calculated as

$$\tau^{\mu} = c^{2} L^{\mu}_{\nu} q^{\nu} \tag{11}$$

where q^{ν} is the heat flux, calculated in the comoving frame, and L^{μ}_{ν} a Lorentz transfromation between the laboratory and each local equilibrium element of the fluid. This idea was firstly set forward, for the equilibrium quantities, by S. Weinberg [8]. This result was obtained by introducing such transformation to relate chaotic and molecular velocities and shows that the heat flux can be consistently defined only in the comoving frame. Because of that, from now on we will consider the hydrodynamic four-velocity as given by Eq. (10) and the molecular velocity v^{μ} will correspond to the chaotic velocity.

In order to solve Eq. (2) the standard Chapman-Enskog method will be used [4, 9]. Thus, the solution is approximated by

$$f = f^{(0)}\left(1 + \phi\left(v^{\mu}\right)\right) \tag{12}$$

where $f^{(0)}$ is the Jüttner equilibrium distribution function which, in the comoving frame, reads [10]

$$f^{(0)} = \frac{n}{4\pi c^{3}} \frac{1}{zK_{2}\left(\frac{1}{z}\right)} \exp^{-\frac{\gamma}{z}}, \tag{13}$$

Here $z = kT/mc^2$ is the relativistic parameter, where T is the local temperature, k the Boltzmann constant, and K_n is the n-th modified Bessel function of the second kind. The solubility conditions imposed on $\phi(\vec{v})$ are given by

$$\int f^{(0)} \epsilon \phi \left(v^{\mu}\right) \begin{pmatrix} m\gamma v^4 \\ m v^{\nu} \end{pmatrix} dv^* = 0 \tag{14}$$

which amounts to restrict the local state variables to be defined through the local equilibrium state. The proposed solution given in Eq. (12) is substituted in Eq. (2). Considering the deviation from the local equilibrium state $\phi(v^{\mu})$ to be a first order quantity, one obtains a linearized first order Boltzmann equation which can be written as

$$v^{\alpha} f^{(0)}_{,\alpha} = f^{(0)} \mathbb{C} \left(\phi \left(v^{\mu}\right)\right) \tag{15}$$

where the linearized collision kernel is given by

$$\mathbb{C}\left(\phi\right) = \int \int \left\{\phi_1' + \phi' - \phi_1 - \phi\right\} f_1^{(0)} \mathcal{F} \sigma \left(\Omega\right) d\Omega dv_1^*. \tag{16}$$

The general solution to equation (15) is given by the sum of the homogeneous solution plus a particular solution, $\phi = \phi_H + \phi_P$. The homogeneous solution is obtained as a linear combination of the collision invariants

$$\mathbb{C} \begin{pmatrix} m v^{\mu} \\ m\gamma v^4 \end{pmatrix} = 0. \tag{17}$$

Existence of the particular solution is guaranteed by imposing an orthogonality condition on the homogeneous solution and the inhomogeneous equation namely,

$$\int \begin{pmatrix} m v^{\mu} \\ m\gamma v^4 \end{pmatrix} v^{\alpha} f^{(0)}_{,\alpha} dv^* = 0, \tag{18}$$

Equations (18) are the relativistic Euler equations obtained through the equilibrium solution [11, 12]. In the absence of external forces, the left hand side of the relativistic Boltzmann equation is written

$$v^{\alpha} f_{,\alpha} = v^{\alpha} \left(\frac{\partial f^{(0)}}{\partial n} n_{,\alpha} + \frac{\partial f^{(0)}}{\partial T} T_{,\alpha} + \frac{\partial f^{(0)}}{\partial u^{\mu}} u^{\mu}_{;\alpha}\right). \tag{19}$$

The next step consists in substituting the derivatives of the Jüttner function and using the Euler equations to write the time derivatives in terms of the gradients. Such equations

constitute a closed set for the state variables. At this point an appropriate representation needs to be chosen and thus we consider n, T and $\mathcal{U}^{\nu}$ as the set of state variables. The Euler equations are written as

$$\dot{n} = -n u^{\alpha}_{;\alpha} \tag{20}$$

$$\dot{u}_{\alpha} = -zc^2 \frac{K_2\left(\frac{1}{z}\right)}{K_3\left(\frac{1}{z}\right)} \left(\frac{n_{,\mu}}{n} + \frac{T_{,\mu}}{T}\right) h^{\mu}_{\alpha} \tag{21}$$

$$\dot{T} = -\frac{T\beta}{nC_n k} u^{\alpha}_{;\alpha} \tag{22}$$

where $\dot{()} = u^{\nu}\,()_{;\nu}$. Here the gradient of the hydrostatic pressure has been written in terms of the gradients of the number density and temperature by using an ideal gas equation of state which can be easily shown to hold for dilute special relativistic gases. We have also used the relation

$$\frac{p}{zc^2\left(\frac{n\epsilon}{c^2} + \frac{p}{c^2}\right)} = \frac{K_2\left(\frac{1}{z}\right)}{K_3\left(\frac{1}{z}\right)} \tag{23}$$

which can be verified by calculating ε and p from the local equilibrium distribution function. After a somewhat tedieous but straightforward algebraic manupulation one can write Eq. (15) as follows

$$v^{\beta} h^{\alpha}_{\beta} \left\{ \frac{n_{,\alpha}}{n}\left(1 - \gamma \frac{K_2\left(\frac{1}{z}\right)}{K_3\left(\frac{1}{z}\right)}\right) + \frac{T_{,\alpha}}{T}\left(1 + \frac{\gamma}{z} - \gamma \frac{K_2\left(\frac{1}{z}\right)}{K_3\left(\frac{1}{z}\right)} - \frac{K_3\left(\frac{1}{z}\right)}{zK_2\left(\frac{1}{z}\right)}\right)\right\} = \mathbb{C}\left(\phi\right) \tag{24}$$

where the term proportional to the hydrodynamic velocity gradient does not arise since the calculations are performed in a comoving frame.

The solution to Eq. (24) is given by

$$\phi = \mathcal{A}\left(\gamma\right) v^{\beta} h^{\alpha}_{\beta} \frac{T_{,\alpha}}{T} + \mathcal{B}\left(\gamma\right) v^{\beta} h^{\alpha}_{\beta} \frac{n_{,\alpha}}{n} + \alpha + \tilde{\alpha}_{\nu} v^{\nu}. \tag{25}$$

The first two terms are the particular solution and the last two terms correspond to the solution of the homogeneous equation. The solubility conditions are thus written as

$$\int \left(\mathcal{A}(\gamma) v^{\beta} h^{\alpha}_{\beta} \frac{T_{,\alpha}}{T} + \mathcal{B}(\gamma) v^{\beta} h^{\alpha}_{\beta} \frac{n_{,\alpha}}{n} + \alpha + \tilde{\alpha}_{\nu} v^{\nu}\right) \psi f^{(0)} dv^* = 0 \tag{26}$$

where $\psi = mv^{\mu}, m\gamma^2$. These conditions imply, as shown in Appendix A, that the constant α vanishes and $\tilde{\alpha}_{\beta}$ is proportional to both $h^{\alpha}_{\beta} n_{,\alpha}$ and $h^{\alpha}_{\beta} T_{,\alpha}$ so that Eq. (25) reads

$$\phi = \mathcal{A}(\gamma) v^{\beta} h^{\alpha}_{\beta} \frac{T_{,\alpha}}{T} + \mathcal{B}(\gamma) v^{\beta} h^{\alpha}_{\beta} \frac{n_{,\alpha}}{n} \tag{27}$$

In Eq. (27), since $n_{,\alpha}$ and $T_{,\alpha}$ are considered independent forces, $\mathcal{A}(\gamma)$ and $\mathcal{B}(\gamma)$ are subject to the constraints

$$\int \mathcal{A}(\gamma)\gamma^2\omega^2 f^{(0)} dv^* = 0, \tag{28}$$

$$\int \mathcal{B}(\gamma)\gamma^2\omega^2 f^{(0)} dv^* = 0. \tag{29}$$

To take full advantage of the fact that the unknowns $\mathcal{A}$ and $\mathcal{B}$ are functions of γ, we perform all integrals in such variable using the relation

$$dv^* = 4\pi c^3\sqrt{\gamma^2-1}d\gamma \tag{30}$$

which is obtained in Appendix B.

III. EXPANSION IN ORTHOGONAL POLYNOMIALS

By substituting Eq. (27) in Eq. (24), the mathematical problem set up in the previous section yields two independent integral equations given by

$$v^\beta h^\alpha_\beta \left\{1 + \frac{\gamma}{z} - \gamma\frac{K_2\left(\frac{1}{z}\right)}{K_3\left(\frac{1}{z}\right)} - \frac{K_3\left(\frac{1}{z}\right)}{zK_2\left(\frac{1}{z}\right)}\right\} = \mathbb{C}(\mathcal{A}(\gamma)v^\beta h^\alpha_\beta) \tag{31}$$

and

$$v^\beta h^\alpha_\beta \left\{1 - \gamma\frac{K_2\left(\frac{1}{z}\right)}{K_3\left(\frac{1}{z}\right)}\right\} = \mathbb{C}(\mathcal{B}(\gamma)v^\beta h^\alpha_\beta) \tag{32}$$

subject to the contraints given by Eqs. (28) and (29) respectively. The unkown coefficients $\mathcal{A}$ and $\mathcal{B}$ are written in terms of orthogonal polynomials in γ

$$\mathcal{A}(\gamma) = \sum_{n=0}^{\infty} a_n\mathcal{L}_n(\gamma) \tag{33}$$

$$\mathcal{B}(\gamma) = \sum_{n=0}^{\infty} b_n\mathcal{L}_n(\gamma) \tag{34}$$

which satisfy the orthogonality condition

$$\int \mathcal{L}_n(\gamma)\mathcal{L}_m(\gamma)p(\gamma)d\gamma = \delta_{nm}, \tag{35}$$

where the weight function $p(\gamma) = \exp^{-\frac{\gamma}{z}}(\gamma^2-1)^{3/2}$. In the case where the hydrodynamic velocity is the one given in Eq. (10), these polyomials, the first two of which are obtained in Appendix C, are related to Kelly's set [4, 13] $R^n_{\frac{3}{2}}$ by the relation

$$R^n_{\frac{3}{2}} = \sqrt{3K_2\left(\frac{1}{z}\right)}z\mathcal{L}_n(\gamma) \tag{36}$$

In terms of the polynomials $\mathcal{L}$, the subsidiary conditions can be written as

$$\sum_{n=0}^{\infty} a_n \int \mathcal{L}_n(\gamma) p(\gamma) d\gamma = 0, \tag{37}$$

$$\sum_{n=0}^{\infty} b_n \int \mathcal{L}_n(\gamma) p(\gamma) d\gamma = 0, \tag{38}$$

Since $\mathcal{L}_0(\gamma)$ is constant, we have that $a_0 = b_0 = 0$ and thus

$$\mathcal{A}(\gamma) = \sum_{n=1}^{\infty} a_n \mathcal{L}_n(\gamma), \tag{39}$$

$$\mathcal{B}(\gamma) = \sum_{n=1}^{\infty} b_n \mathcal{L}_n(\gamma). \tag{40}$$

The heat flux in the Chapman-Enskog approximation, as clearly stated in Ref. [3], is given by the average of the chaotic kinetic energy flux, a definition that encompasses the physical conception of heat since the early developments of kinetic theory [14, 15]. Since in this work the molecular and chaotic velocities coincide, we can write

$$q^{\mu} = mc^2 h^{\mu}_{\nu} \int \gamma v^{\nu} f^{(1)} d^*v \tag{41}$$

or, subtituting Eq. (41)

$$q^{\mu} = \frac{nm}{4\pi c z K_2(\frac{1}{z})} \left[I^{\mu}_{(a)} + I^{\mu}_{(b)} \right]. \tag{42}$$

where

$$I^{\mu}_{(a)} = h^{\mu}_{\nu} h^{\alpha}_{\beta} \frac{T_{,\alpha}}{T} \int \gamma v^{\nu} v^{\beta} \mathcal{A}(\gamma) e^{-\frac{\gamma}{z}} d^*v \tag{43}$$

$$I^{\mu}_{(b)} = h^{\mu}_{\nu} h^{\alpha}_{\beta} \frac{n_{,\alpha}}{n} \int \gamma v^{\nu} v^{\beta} \mathcal{B}(\gamma) e^{-\frac{\gamma}{z}} d^*v. \tag{44}$$

Notice that in both integrals only the $\nu, \beta = 1, 2, 3$ terms survive and from them, all $\nu \neq \beta$ ones also vanish because the integrands are odd in the three-velocity. Thus, introducing Eqs. (39) and (40), the integrals read

$$I^{\mu}_{(a)} = \frac{4\pi c^5}{3} h^{\mu\alpha} \frac{T_{,\alpha}}{T} \sum_{n=1}^{\infty} a_n \int \gamma \mathcal{L}_n(\gamma) p(\gamma) d\gamma \tag{45}$$

$$I^{\mu}_{(b)} = \frac{4\pi c^5}{3} h^{\mu\alpha} \frac{n_{,\alpha}}{n} \sum_{n=1}^{\infty} b_n \int \gamma \mathcal{L}_n(\gamma) p(\gamma) d\gamma. \tag{46}$$

As shown in Appendix C, we can write $\gamma = c_0 \mathcal{L}_0(\gamma) + c_1 \mathcal{L}_1(\gamma)$, with $c_1 = \sqrt{3g(z)}z$ where

$$g(z) = 5zK_3\left(\frac{1}{z}\right) + K_2\left(\frac{1}{z}\right) - \frac{K_3\left(\frac{1}{z}\right)^2}{K_2\left(\frac{1}{z}\right)} \tag{47}$$

Using Eqs. (45-47) in the heat flux given by Eq. (42) we obtain that

$$q^{\mu} = -h^{\mu\alpha}\left[L_T \frac{T_{,\alpha}}{T} + L_n \frac{n_{,\alpha}}{n}\right] \tag{48}$$

where the coefficients appearing in Eq. (48) are defined as

$$L_T = -\frac{nmc^4 c_1}{3zK_2(\frac{1}{z})} a_1 \tag{49}$$

$$L_n = -\frac{nmc^4 c_1}{3zK_2(\frac{1}{z})} b_1 . \tag{50}$$

IV. SOLUTION OF THE INTEGRAL EQUATIONS

The coefficients $a_1$and b_1, in terms of which the coefficients in Eqs. (49) and (50) are given, have to be obtained from the solution of the integral equations (31) and (32). In this section we outline such calculation to a first approximation. The variational method used is the standard one as described in detail in Ref. [16].

First we notice that the integral equations (31-32) can be written as

$$-\sqrt{3g\left(z\right)}z\left(\frac{K_2\left(\frac{1}{z}\right)}{K_3\left(\frac{1}{z}\right)} - \frac{1}{z}\right) f^{(0)} v^{\beta} h^{\alpha}_{\beta} \mathcal{L}_1(\gamma) = h^{\alpha}_{\beta} \sum_{n=1}^{\infty} a_n \mathbb{C}(\mathcal{L}_n(\gamma) v^{\beta} f^{(0)}) \tag{51}$$

$$-\frac{K_2\left(\frac{1}{z}\right)}{K_3\left(\frac{1}{z}\right)} \sqrt{3g\left(z\right)} z f^{(0)} v^{\beta} h^{\alpha}_{\beta} \mathcal{L}_1(\gamma) = h^{\alpha}_{\beta} \sum_{n=1}^{\infty} b_n \mathbb{C}(\mathcal{L}_n(\gamma) v^{\beta} f^{(0)}) \tag{52}$$

such that both have a similar structure. Indeed, since the dependence on γ on both is the same, the procedure only needs to be carried out once for one of the equations and the solution for the other one can be readily inferred by adjusting the dependence on the parameter z. This similarity is consistent with the calculation in Ref. [4] where only one integral equation needs to be solved for the coefficient of a generalized thermodynamic force which includes contributions from ∇T and ∇p in a single term.

Following the presciption mentioned above, we will only deal with Eq. (52). Multiplying it by $\mathcal{L}_m(\gamma) v_{\nu} h^{\nu}_{\alpha}$ and integrating on both sides

$$-\frac{\sqrt{3g}}{K_3\left(\frac{1}{z}\right)} \frac{n}{4\pi c^3} \int h^{\nu}_{\beta} e^{-\frac{\gamma}{z}} v_{\nu} v^{\beta} \mathcal{L}_1 \mathcal{L}_m d^*v = h^{\nu}_{\beta} \sum_{n=1}^{\infty} b_n \int \mathcal{L}_m v_{\nu} \mathbb{C}(\mathcal{L}_n v^{\beta} f^{(0)}) d^*v \tag{53}$$

where we have omitted the z and γ dependences to short notation. For the integral on left hand side, using that $e^{-\frac{\gamma}{z}} h^{\nu}_{\beta} v_{\nu} v^{\beta} = 4\pi c^5 p(\gamma) d\gamma$, we have

$$\int e^{-\frac{\gamma}{z}} h^{\nu}_{\beta} v_{\nu} v^{\beta} \mathcal{L}_1(\gamma) \mathcal{L}_m(\gamma) d^*v = 4\pi c^5 \delta_{1m} \tag{54}$$

and thus, defining the collision brakett in the standard way

$$[G,H]=-\frac{1}{n^2}\int G_\alpha\cdot\left[H_{\alpha 1}^{'}+H_{\alpha}^{'}-H_{\alpha 1}-H_\alpha\right]f^{(0)}f_1^{(0)}\mathcal{F}\sigma\left(\Omega\right)d\Omega dv_1^*dv^* \tag{55}$$

in this equation $G_\alpha=G_\alpha(v_\beta)$ and we have used $H_{\alpha 1}^{'}$ to denote $H_\alpha\left(v_{\beta 1}^{'}\right)$. The integral equations can be written as

$$h_\beta^\nu\sum_{n=1}^{\infty}a_n\left[\mathcal{L}_m(\gamma)v_\nu,\mathcal{L}_n(\gamma)v^\beta\right]=\left(\frac{K_2\left(\frac{1}{z}\right)}{K_3\left(\frac{1}{z}\right)}-\frac{1}{z}\right)\frac{c^2\sqrt{3g\left(z\right)}}{nK_2\left(\frac{1}{z}\right)}\delta_{1m} \tag{56}$$

$$h_\beta^\nu\sum_{n=1}^{\infty}b_n\left[\mathcal{L}_m(\gamma)v_\nu,\mathcal{L}_n(\gamma)v^\beta\right]=c^2\frac{\sqrt{3g\left(z\right)}}{nK_3\left(\frac{1}{z}\right)}\delta_{1m} \tag{57}$$

By following the standard variational method [16], the first approximation for a_1 and b_1 can be shown to be given by

$$a_1=\left(\frac{K_2\left(\frac{1}{z}\right)}{K_3\left(\frac{1}{z}\right)}-\frac{1}{z}\right)\frac{\sqrt{3g\left(z\right)}}{nK_2\left(\frac{1}{z}\right)}c^2\left\{h_\beta^\nu\left[\mathcal{L}_1(\gamma)v_\nu,\mathcal{L}_1(\gamma)v^\beta\right]\right\}^{-1} \tag{58}$$

$$b_1=c^2\frac{\sqrt{3g\left(z\right)}}{nK_3\left(\frac{1}{z}\right)}\left\{h_\beta^\nu\left[\mathcal{L}_1(\gamma)v_\nu,\mathcal{L}_1(\gamma)v^\beta\right]\right\}^{-1} \tag{59}$$

Thus, in order to calculate the coefficients a_1 and b_1 to this level of approximation only one collision integral needs to be calculated namely, $\left[\mathcal{L}_1(\gamma)v_\nu,\mathcal{L}_1(\gamma)v^\beta\right]$. In order to calculate such brakett, the well known identity

$$\begin{aligned}[G_\alpha,H_\beta]=\frac{1}{4n^2}\int&\left[G_\alpha\left(v^{\mu'}\right)+G_\alpha\left(v_1^{\mu'}\right)-G_\alpha\left(v^{\mu}\right)-G_\alpha\left(v_1^{\mu}\right)\right]\cdot\\&\left[H_\beta\left(v_1^{\mu'}\right)+H_\beta\left(v^{\mu'}\right)-H_\beta\left(v_1^{\mu}\right)-H_\beta\left(v^{\mu}\right)\right]f^{(0)}f_1^{(0)}\mathcal{F}\sigma\left(\Omega\right)d\Omega dv_1^*dv^*,\end{aligned} \tag{60}$$

will be used. Using also the momentum conservation law for collisions and after several algebraic steps one can show that

$$h_\beta^\nu\left[\mathcal{L}_1(\gamma)v_\nu,\mathcal{L}_1(\gamma)v^\beta\right]=-\frac{\mathcal{I}_1-\mathcal{I}_2}{3g\left(z\right)z^2n^2m^4c^5} \tag{61}$$

where the integrals $\mathcal{I}_1$ and $\mathcal{I}_2$ are defined as

$$\mathcal{I}_1=m^4c^7\int\int\gamma^2\left[\left(\gamma^2\right)_1'+\left(\gamma^2\right)'-\left(\gamma^2\right)_1-\gamma^2\right]f^{(0)}f_1^{(0)}\mathcal{F}\sigma d\Omega dv_1^*dv^* \tag{62}$$

$$\mathcal{I}_2=-m^4c^5\int\int\gamma v^\beta\left[\left(\gamma v_\beta\right)_1'+\left(\gamma v_\beta\right)'-\left(\gamma v_\beta\right)_1-\gamma v_\beta\right]f^{(0)}f_1^{(0)}\mathcal{F}\sigma d\Omega dv_1^*dv^*. \tag{63}$$

Thus, the general expressions for the coefficients L_T and L_n are given by

$$L_T = \frac{3k^2T^2n^2m^3c^7}{\mathcal{I}_1 - \mathcal{I}_2}\left(\frac{K_2\left(\frac{1}{z}\right)}{K_3\left(\frac{1}{z}\right)} - \frac{1}{z}\right)\left(\frac{g\left(z\right)}{K_2\left(\frac{1}{z}\right)}\right)^2 \tag{64}$$

$$L_n = -\frac{3k^2T^2n^2m^3c^7}{\mathcal{I}_1 - \mathcal{I}_2}\frac{\left(g\left(z\right)\right)^2}{K_3\left(\frac{1}{z}\right)K_2\left(\frac{1}{z}\right)}. \tag{65}$$

In order to evaluate the integrals in $\mathcal{I}_1$ and $\mathcal{I}_2$, a particular collision model needs to be proposed. In the next section the simplest collision model, namely a constant cross section, will be assumed in order to obtain expression for the coefficients and asess their relative magnitude.

V. CONSTANT SCATTERING CROSS SECTION

The simplest model that one might consider in order to calculate collision integrals consists in assuming a constant cross section. The details of the calculations for such model can be found in Ref. [4]. Here, we will only quote the final results for the integrals in Eqs. (62) and (63)

$$\mathcal{I}_1 = -\frac{64\pi n^2k^6T^6\sigma}{m^2c^4\left(K_2\left(\frac{1}{z}\right)\right)^2}\left(2K_2\left(\frac{2}{z}\right) + \frac{1}{z}K_3\left(\frac{2}{z}\right)\right) \tag{66}$$

$$\mathcal{I}_2 = \frac{64\pi n^2k^6T^6\sigma}{m^2c^4\left(K_2\left(\frac{1}{z}\right)\right)^2}\left(\frac{4}{z}K_3\left(\frac{2}{z}\right) + \frac{1}{z^2}K_2\left(\frac{2}{z}\right)\right) \tag{67}$$

where σ is the constant scattering cross section. Subtituting these expressions in Eqs. (64) and (65) one obtains

$$L_T = -\frac{3ckT}{64\pi\sigma}\left(z\frac{K_2\left(\frac{1}{z}\right)}{K_3\left(\frac{1}{z}\right)} - 1\right)\frac{\left(K_2\left(\frac{1}{z}\right)\left[\frac{1}{z} + 5\mathcal{G}\left(\frac{1}{z}\right) - \frac{1}{z}\mathcal{G}\left(\frac{1}{z}\right)^2\right]\right)^2}{z^4\left(\frac{5}{z}K_3\left(\frac{2}{z}\right) + \left(\frac{1}{z^2} + 2\right)K_2\left(\frac{2}{z}\right)\right)} \tag{68}$$

$$L_n = -\frac{3ckT}{64\pi\sigma}\frac{K_2\left(\frac{1}{z}\right)\left(K_2\left(\frac{1}{z}\right)\left[\frac{1}{z} + 5\mathcal{G}\left(\frac{1}{z}\right) - \frac{1}{z}\mathcal{G}\left(\frac{1}{z}\right)^2\right]\right)^2}{z^3K_3\left(\frac{1}{z}\right)\left(\frac{5}{z}K_3\left(\frac{2}{z}\right) + \left(\frac{1}{z^2} + 2\right)K_2\left(\frac{2}{z}\right)\right)} \tag{69}$$

which are to some extent approximations to the coefficients in appearing in the heat flux constitutve equation in the n, T, $\mathcal{U}^\nu$ representation. In the limit of small relativistic paramenter z one recovers the non-relativistic values

$$L_T \sim L_{TNR}\left\{1 - \frac{3}{16}z + ...\right\} \tag{70}$$

$$L_n \sim -L_{TNR}\left\{ z - \frac{27}{16} z^2 + ... \right\} \tag{71}$$

where $L_{TNR} = (75mc^3/256\sqrt{\pi}\sigma)z^{3/2}$ is the usual non-relativistic thermal conductivity for hard spheres divided by T. Notice that indeed, as z goes to zero $L_T \to L_{TNR}$ and $L_n \to 0$. In Fig. 1 the magnitudes of both coefficients, normalized to L_{TNR} are shown as functions of z.

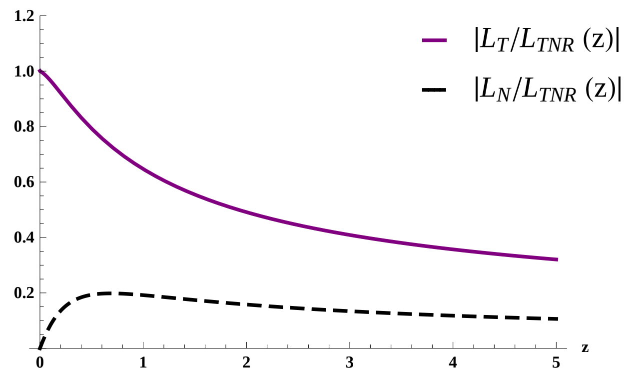

Figure 1: The magnitudes of both coefficients, normalized to L_{TNR} are shown as functions of z.

VI. DISCUSSION AND FINAL REMARKS

In the previous section the transport coefficients involved in the constitutive equation for the heat flux in terms of the forces ∇T and ∇n have been obtained. The calculation has been performed in the fluid's local comoving frame, that is in terms of the chaotic or pecular velocity. The Chapman-Enskog solution method of Boltzmann's relativistic equation in this representation leads to two independent integral equations. Once the solution is written in terms of a suitable orthogonal set of polynomials, both equations have the same structure and the solution is finally obtained using the standard variational method. By assuming a very simple constant cross section model to calculate the collision integrals, the coefficients can be studied. The results are shown in Fig. 1. It can be clearly seen that for small values of z the non-relativistic limit is verified. For larger values of the relativistic parameter the coefficient L_n increases and has the same order of magnitude of L_T. That is, if the gradients of the system are of the same order of magnitude the relativistic effect $q \propto \nabla n$ is as important as the usual Fourier effect.

In previous work, Cercigniani & Kremer obtained the constitutive equation for the heat flux in a relativistic neutral gas in a different way. The definition they used for the heat flux is the one obtained from the phenomenology using Eckart's decomposition in an arbitary frame. However, in the comoving frame both definitions, the one given here in Eq. (41) and the one they consider namely,

$$q^\alpha = h^\alpha_\beta \mathcal{U}_\gamma c \int p^\beta p^\gamma f \frac{d^3p}{p^4} \tag{72}$$

where p^α is the four-momentum, are the same. However, after the calculation of the heat flux from the Chapman-Enskog solution of the Boltzman equation, in Ref. [4] the heat flux is obtained in terms of a relativistic thermal force which includes both the Fourier term as well as the relativistic correction which they leave in terms of the gradient of the hydrostatic pressure. Their result is

$$q^\alpha = \lambda h^{\alpha\beta} \left[T_{,\beta} - \frac{T}{nh_e} p_{,\beta} \right] \tag{73}$$

where

$$h_E = \epsilon + \frac{p}{n} = \frac{p}{nz} \frac{K_3\left(\frac{1}{z}\right)}{K_2\left(\frac{1}{z}\right)} \tag{74}$$

and λ is given by

$$\lambda = -\frac{3kp^2m^2c^5\left(\frac{1}{z} + 5\mathcal{G}\left(\frac{1}{z}\right) - \frac{1}{z}\mathcal{G}\left(\frac{1}{z}\right)^2\right)^2}{I_1 - c^2 I_2} \tag{75}$$

with the integrals I_1 and I_2 defined as

$$I_1 = \mathcal{U}_\alpha \mathcal{U}_\beta \mathcal{U}_\gamma \mathcal{U}_\delta \times$$
$$\iint p^\alpha p^\beta \left\{ \left(p^\gamma p^\delta\right)_1' + \left(p^\gamma p^\delta\right)' - \left(p^\gamma p^\delta\right)_1 - p^\gamma p^\delta \right\} f_c^{(0)} f_{c1}^{(0)} F \sigma d\Omega \frac{d^3p_1}{p_1^4} \frac{d^3p}{p^4}$$

$$I_2 = \mathcal{U}_\alpha \mathcal{U}_\gamma \times$$
$$\iint p^\alpha p^\beta \left\{ (p^\gamma p_\beta)_1' + (p^\gamma p_\beta)' - (p^\gamma p_\beta)_1 - p^\gamma p_\beta \right\} f_c^{(0)} f_{c1}^{(0)} F \sigma d\Omega \frac{d^3p_1}{p_1^4} \frac{d^3p}{p^4}$$

The invariant flux in the expressions above is related with the one in Eq. (4) by $F = m^2 c \mathcal{F}$ and also the distribution function in Ref. [4] differs by a factor from the one used in this work: $f_c = \frac{1}{m^3} f$.

In order to compare Eq. (73) with our results the hydrostatic pressure can be written in terms of ∇n and ∇T by means of the ideal gas equation of state. Also the hydrodynamic

velocity has to be substituted by the one given in Eq. (10), which accounts to writting Cercigniani's result [4] in the comoving frame. After these changes have been introduced, the integration variable changed to γ and the different signature is taken into account, one can readily verify that $I_1 = \mathcal{I}_1$ and $I_2 = c^{-2}\mathcal{I}_2$ and that the constitutive equation in Eq. (73) can be written as

$$q^\alpha = h^{\alpha\beta}\left[\lambda T\left(1 - z\frac{K_2\left(\frac{1}{z}\right)}{K_3\left(\frac{1}{z}\right)}\right)\frac{T_{,\beta}}{T} - \lambda T z\frac{K_2\left(\frac{1}{z}\right)}{K_3\left(\frac{1}{z}\right)}\left(\frac{n_{,\beta}}{n}\right)\right] \tag{76}$$

which has the same structure of Eq. (48). Subtituting the value for λ obtained in Ref. [4] which can be written as

$$\lambda = -\frac{3kn^2m^4c^9}{I_1 - c^2I_2}\left(\frac{g(z)}{K_2\left(\frac{1}{z}\right)}\right)^2 \tag{77}$$

yields

$$\lambda T\left(1 - z\frac{K_2\left(\frac{1}{z}\right)}{K_3\left(\frac{1}{z}\right)}\right) = \left(\frac{K_2\left(\frac{1}{z}\right)}{K_3\left(\frac{1}{z}\right)} - \frac{1}{z}\right)\frac{3n^2k^2T^2m^3c^7}{I_1 - c^2I_2}\left(\frac{g(z)}{K_2\left(\frac{1}{z}\right)}\right)^2 \tag{78}$$

$$\lambda T z\frac{K_2\left(\frac{1}{z}\right)}{K_3\left(\frac{1}{z}\right)} = -\frac{3n^2k^2T^2m^3c^7}{I_1 - c^2I_2}\frac{(g(z))^2}{K_3\left(\frac{1}{z}\right)K_2\left(\frac{1}{z}\right)} \tag{79}$$

which are precisely the coefficients in Eqs. (64) and (65). Thus, one concludes that both calculations are consistent in the comoving frame.

As mentioned above, the constitutive equation obtained in this work is equivalent to the one obtained in Ref. [4] only in the comoving frame since in that case the definitions for the heat flux are identical. In this work we follow the ideas set forth in Ref. [3] where the heat flux is a local quantity which only makes sense in the comoving frame where the molecular velocity coincides with the peculiar velocity. However, the term appearing in the stress-energy tensor and that will ultimately impact the transport equations is in our case given by $\tau^\mu = c^2 L^\mu_\nu q^\nu$ with q^ν given by Eq. (41) while in the traditional calculations what is obtained is $\tau^\mu = q^\nu$ with q^ν given by Eq. (72). This difference is discussed in Ref. [3] and its implications will be explored in future work.

Acknowledgments

The authors wish to thank Dr. A. Sandoval-Villalbazo, Prof. García-Colín and Prof. G. Medeiros-Kremer for valuable comments and suggestions. This work was partially supported by PROMEP grant UAM-PTC-142.

Appendix A

In this appendix we show that Eq. (27) is valid in view of the subsidiary conditions given in Eq. (26). Equation (26) can be written as

$$\int \left(\mathcal{A}(\gamma) v^\ell \frac{T_{,\ell}}{T} + \mathcal{B}(\gamma) v^\ell \frac{n_{,\ell}}{n} + \alpha + \tilde{\alpha}_\ell v^\ell + \tilde{\alpha}_4 v^4 \right) \psi f^{(0)} dv^* = 0, \tag{80}$$

where we have separated spatial and temporal terms in the contraction $\tilde{\alpha}_\nu v^\nu$ since they have opposite parity.

For $\psi = v^4 = \gamma c$ the first, second and forth terms yield odd integrands and thus

$$\int \left(\alpha + \tilde{\alpha}_4 v^4 \right) \gamma f^{(0)} dv^* = 0 \tag{81}$$

For $\psi = v^\ell$ with $\ell = 1$, 2, 3 the third and fifth terms do not contribute because of their odd parity and we obtain

$$\int \left(\mathcal{A}(\gamma) \frac{T_{,\ell}}{T} + \mathcal{B}(\gamma) \frac{n_{,\ell}}{n} + \tilde{\alpha}_\ell \right) v^\ell v^k f^{(0)} dv^* = 0, \tag{82}$$

which vanishes for $\ell \neq k$. Thus

$$\int \left(\mathcal{A} \frac{T_{,\ell}}{T} + \mathcal{B} \frac{n_{,\ell}}{n} + \tilde{\alpha}_\ell \right) v^\ell v^k f^{(0)} dv^* = \frac{\delta^{k\ell}}{3} \int \left(\mathcal{A} \frac{T_{,\ell}}{T} + \mathcal{B} \frac{n_{,\ell}}{n} + \tilde{\alpha}_\ell \right) \gamma^2 w^2 f^{(0)} dv^*, \tag{83}$$

and thus the condition in Eq. (82) reduces to

$$\int \left(\mathcal{A}(\gamma) \frac{T_{,\ell}}{T} + \mathcal{B}(\gamma) \frac{n_{,\ell}}{n} + \tilde{\alpha}_\ell \right) \gamma^2 w^2 f^{(0)} dv^* = 0. \tag{84}$$

Finally, for $\psi = \gamma^2$ the parity in the terms is the same as for $\psi = v^4$ such that

$$\int \left(\alpha + \tilde{\alpha}_4 v^4 \right) \gamma^2 f^{(0)} dv^* = 0 \tag{85}$$

From equation (82) we have that

$$\tilde{\alpha}_k = -\frac{T_{,k}}{T} \frac{\int \mathcal{A}(\gamma) \gamma^2 \omega^2 f^{(0)} dv^*}{\int \gamma^2 \omega^2 f^{(0)} dv^*} - \frac{n_{,k}}{n} \frac{\int \mathcal{B}(\gamma) \gamma^2 \omega^2 f^{(0)} dv^*}{\int \gamma^2 \omega^2 f^{(0)} dv^*}, \tag{86}$$

such that we can redefine

$$\mathcal{A}(\gamma) v^k \frac{T_{,k}}{T} + \mathcal{B}(\gamma) v^k \frac{n_{,k}}{n} + \tilde{\alpha}_k v^k \longrightarrow a(\gamma) v^k \frac{T_{,k}}{T} + b(\gamma) v^k \frac{n_{,k}}{n} \tag{87}$$

and now the subsidiary condition (82) reads

$$\int \left[\mathcal{A}(\gamma) \frac{T_{,k}}{T} + \mathcal{B}(\gamma) \frac{n_{,k}}{n} \right] \gamma^2 \omega^2 f^{(0)} dv^* = 0. \tag{88}$$

Equations (81) and (85) can be written as an homogeneous system for $(\alpha, \tilde{\alpha_4})$:

$$\begin{aligned} \alpha g_{11} + \tilde{\alpha_4} g_{12} &= 0, \\ \alpha g_{21} + \tilde{\alpha_4} g_{22} &= 0, \end{aligned} \tag{89}$$

where

$$g_{11} = \int \gamma f^{(0)} dv^*, \quad g_{12} = g_{21} = \int \gamma^2 c f^{(0)} dv^*, \qquad g_{22} = \int \gamma^3 c^2 f^{(0)} dv^*, \tag{90}$$

such that the determinant does not vanish and the solution is the trival one $\alpha = \tilde{\alpha_4} = 0$. Putting toghether these two results, we can conclude that the proposed solution consistent with the subsiadiary conditions is the one given by Eq. (27) where the condition in Eq. (82) still needs to be enforced. Considering both forces $T_{,k}$ and $n_{,k}$ as independent forces, this requirement is written as two separate conditions in Eqs. (28) and (29).

Appendix B

In this appendix, the identity $dv^* = 4\pi c^3 \sqrt{\gamma^2 - 1} d\gamma$, which is used in several parts of the work, will be obtained. The invariant volume element in velocity space is given by [4, 5]

$$dv^* = c\frac{dv^3}{v^4} = \frac{dv^3}{\gamma} \tag{91}$$

or, in terms of the three-velocity $\vec{w}$

$$dv^* = \det[J] \frac{d^3 w}{\gamma} \tag{92}$$

where the Jacobian matrix has components $J_{ij} = \partial v_i / \partial w_j$ which can be shown to be given by

$$J_{ij} = \gamma \left(\delta_{ij} + \gamma^2 \frac{w_i w_j}{c^2} \right) \tag{93}$$

Using the identity $\det[\delta_{ij} + A_i B_j] = 1 + A_i B^i$

$$\det[J] = \gamma^3 \left(1 + \gamma^2 \frac{w^2}{c^2} \right) = \gamma^5 \tag{94}$$

and thus, introducing spherical coordinates for $\vec{w}$ and that $w^2 dw = \frac{c^3}{\gamma^4} \sqrt{\gamma^2 - 1} d\gamma$ we finally obtain

$$dv^* = 4\pi c^3 \sqrt{\gamma^2 - 1} d\gamma \tag{95}$$

Appendix C

The orthogonal polynomials are obtained using the standard Gram-Schmidt procedure [17]. The proposed polynomials are $\mathcal{L}_n(\gamma) = a_{0n} + a_{1n}\gamma + a_{2n}\gamma^2 + \cdots + a_{nn}\gamma^n$ which are required to satisfy the orthonormality condition

$$\int \mathcal{L}_n(\gamma)\mathcal{L}_m(\gamma)p(\gamma)d\gamma = \delta_{mn} \tag{96}$$

where $p(\gamma) = \exp^{-\frac{\gamma}{z}}(\gamma^2-1)^{3/2}$. For $n=0$ we have $\mathcal{L}_0(\gamma) = a_{00}$ and the orthonormality condition (96) yields

$$\mathcal{L}_0(\gamma) = \frac{1}{\sqrt{3}zK_2(\frac{1}{z})^{\frac{1}{2}}}. \tag{97}$$

For $n=1$ we have $\mathcal{L}_1(\gamma) = a_{01} + a_{11}\gamma$ and equation (96) yields

$$a_{01} = -\frac{K_3\left(\frac{1}{z}\right)}{K_2\left(\frac{1}{z}\right)}\frac{1}{\sqrt{3}z}\left[5zK_3\left(\frac{1}{z}\right) + K_2\left(\frac{1}{z}\right) - \frac{K_3\left(\frac{1}{z}\right)^2}{K_2\left(\frac{1}{z}\right)}\right]^{-\frac{1}{2}} \tag{98}$$

$$a_{11} = \frac{1}{\sqrt{3}z}\left[5zK_3\left(\frac{1}{z}\right) + K_2\left(\frac{1}{z}\right) - \frac{K_3\left(\frac{1}{z}\right)^2}{K_2\left(\frac{1}{z}\right)}\right]^{-\frac{1}{2}} \tag{99}$$

if we define $g(z) = 5zK_3\left(\frac{1}{z}\right) + K_2\left(\frac{1}{z}\right) - \frac{K_3\left(\frac{1}{z}\right)^2}{K_2\left(\frac{1}{z}\right)}$

$$\mathcal{L}_1(\gamma) = \frac{1}{\sqrt{3g(z)}z}\left[-\frac{K_3\left(\frac{1}{z}\right)}{K_2\left(\frac{1}{z}\right)} + \gamma\right]. \tag{100}$$

We only need these two polynomials in order to calculate the heat flux. Additionally, the coefficients in $\gamma = c_0\mathcal{L}_0 + c_1\mathcal{L}_1$ are introduced in Sect. III. A simple way of obtaining them is to solve Eq. (100) for γ and use Eq. (97) which yields

$$\gamma = \sqrt{3g(z)}z\mathcal{L}_1(\gamma) + \sqrt{3}z\frac{K_3\left(\frac{1}{z}\right)}{\sqrt{K_2\left(\frac{1}{z}\right)}}\mathcal{L}_0 \tag{101}$$

from which $c_0 = \sqrt{3}z\frac{K_3\left(\frac{1}{z}\right)}{\sqrt{K_2\left(\frac{1}{z}\right)}}$ and $c_1 = \sqrt{3g(z)}z$.

[1] A. L. García-Perciante, A. Sandoval-Villalbazo, J. Non-Newtonian Fluid Mech. **165**, 1024-1028 (2010).

[2] W. A. Hiscock and L. Lindblom, Phys. Rev. D **31**, 725 (1985).

[3] A. L. García-Perciante, A. Sandoval-Villalbazo, L. S. Garcia-Colin; arXiv:1007.2815

[4] C. Cercignani and G. Medeiros Kremer; *The Relativistic Boltzmann Equation: Theory and Applications*, Cambridge University Press 3rd Ed., UK (1991).

[5] S. R. de Groot, W. A. van Leeuwen and Ch. van der Wert; *Relativistic Kinetic Theor*y, North Holland Publ. Co., Amsterdam (1980).

[6] W. Israel; J. Math. Phys **4**, 1163 (1963).

[7] C. Eckart, Phys. Rev. **58,** 267 (1940); ibid **58,** 919 (1940).

[8] S. Weinberg; *Gravitation and Cosmology*, Wiley & Sons, USA (1972); Chapter 2.

[9] S. Chapman, T. G. Cowling; *The Mathematical Theory of Non-Uniform Gases,* Cambridge Univ. Press (1970) .

[10] F. Jüttner; Ann. Physik und Chemie **34** (1911) 856.

[11] C. Courant, D. Hilbert; *Methods of Mathematical Physics*, J. Wiley NY (1972).

[12] A. L. Garcia-Perciante, A. Sandoval-Villalbazo, L. S. Garcia-Colin; Physica A **387**, 5073-5079 (2008).

[13] D. C. Kelly; *The kinetic theory of a relativistic gas* (unpublished report), 1963.

[14] R. Calusius, Annalen der Physik **100**, 353-380 (1857).

[15] S. Brush; *The kind of Motion we call Heat*, North Holland, Amsterdam (1986) .

[16] J. O. Hirshfelder, C. F. Curtiss, R. Byron Bird; *The Molecular Theory of Gases and Liquids*, J. Wiley & Sons NY (1954).

[17] G.B. Arfken and H.J. Weber; *Mathematical Methods for Physicists*, Academic Press, 1995.